L33 April 9 Sphere

Wednesday, April 8, 2015 3:20 PM

Given a space X and a base point $x_0 \in X$

$\pi_1(X, x_0)$ = Fundamental Group of X at x_0

$\cup\!\!\!\mid$

$[\alpha]$ loop homotopy class

$\alpha : ([0,1], \{0,1\}) \longrightarrow (X, x_0)$

under homotopy rel $\{0,1\}$

Group structure

$$[\alpha]\cdot[\beta] = [\alpha * \beta]$$

$$1 = [c] \quad \text{where} \quad c : X \longrightarrow \{x_0\}$$

$$[\alpha]^{-1} = [\bar{\alpha}] \qquad \bar{\alpha}(s) = \alpha(1-s)$$

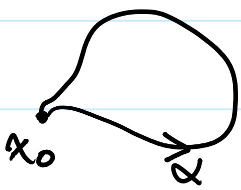
Independent of base point

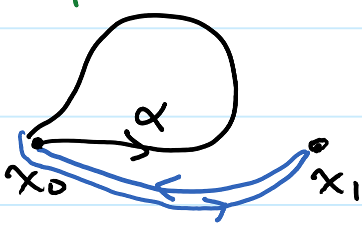
If X is path connected and $x_0, x_1 \in X$

then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic

$\cup\!\!\!\mid$ $\cup\!\!\!\mid$

Loops at x_0 $\longrightarrow$ Loops at x_1





Let $\gamma : [0,1] \longrightarrow X$ be a path with

$\gamma(0) = x_1$, $\gamma(1) = x_0$, from x_1 to x_0

Then $\bar{\gamma}(s) = \gamma(1-s)$ is from x_0 to x_1

Define $\varphi : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ by

$$[\alpha] \longmapsto [\gamma * \alpha * \bar{\gamma}]$$

Need to check

① Well-defined: $\alpha_0 \simeq \alpha_1$ rel $\{0,1\}$

$$\Rightarrow \gamma * \alpha_0 * \bar{\gamma} \simeq \gamma * \alpha_1 * \bar{\gamma} \quad \text{rel } \{0,1\}$$

proved similarly in $[\alpha][\beta] = [\alpha * \beta]$

② Bijection: Obvious inverse by $\bar{\gamma}$ from x_0 to x_1

③ homomorphism: for $[\alpha], [\beta] \in \pi_1(X, x_0)$

$$(\varphi[\alpha]) \cdot (\varphi[\beta]) = \varphi([\alpha][\beta])$$

$$(\gamma * \alpha * \underbrace{\bar{\gamma}) * (\gamma}_{} * \beta * \bar{\gamma}) \qquad \gamma * (\alpha * \beta) * \bar{\gamma}$$

Clearly $\bar{\gamma} * \gamma \simeq c$

Qu. For $[\alpha] \stackrel{\varphi}{\longmapsto} [\gamma * \alpha * \bar{\gamma}]$, is φ independent of choice of γ from x_1 to x_0?

Theorem If $X \simeq Y$ are path connected then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$

The proof is a routine exercise of homotopy.

Simply connected A path connected space X is 1-connected if $\pi_1(X, x_0)$ is trivial

Examples.

(1) X is contractible $\Rightarrow$ X is 1-connected

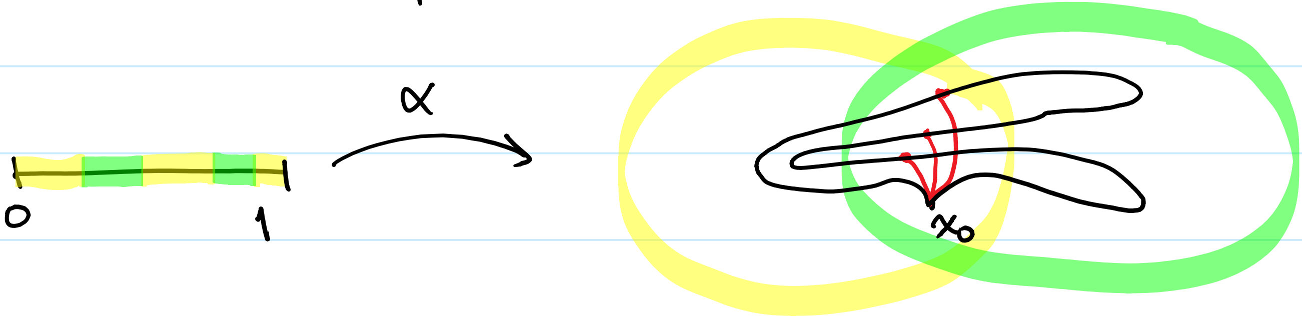
(2) S^2, in general, S^n, $n \geq 2$ is 1-connected.

But they are not contractible

Theorem If $X = A \cup B$ where both A, B are 1-connected open subsets and $A \cap B$ is path connected then X is 1-connected

First, X is clearly path connected.

Choose a base point $x_0 \in A \cap B$



Let $\alpha : [0,1] \longrightarrow X$ Then $\alpha^{-1}(A)$, $\alpha^{-1}(B)$ and $\alpha^{-1}(A \cap B)$ are open sets in $[0,1]$. Thus, they define an open cover by intervals for $[0,1]$. So, there is a finite subcover by intervals, and

a partition $0 = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$

$\alpha_k \equiv \alpha|_{[s_k, s_{k+1}]} : [s_k, s_{k+1}] \longrightarrow A$ or B or $A \cap B$

For each s_j with $\alpha(s_j) \in A, B,$ or $A \cap B$, let γ_j a path from x_0 to $\alpha(s_j)$ in $A, B,$ or $A \cap B$

Then $\alpha \simeq \underbrace{\alpha_0 * \bar{\gamma}_1} * \underbrace{\gamma_1 * \alpha_1 * \bar{\gamma}_2} \cdots * \underbrace{\gamma_j * \alpha_j * \bar{\gamma}_{j+1}} * \cdots * \underbrace{\alpha_{n-1}}$

each is a loop inside A or B based at x_0

$\simeq c_{x_0}$

L33 p4

Wednesday, April 8, 2015 4:09 PM

Van Kampen's Theorem

The previous result is a special easy version of an important theorem. That basically gives $\pi_1(X)$ from $\pi_1(A)$, $\pi_1(B)$, and $\pi_1(A\cap B)$. Note that at the end of the proof,

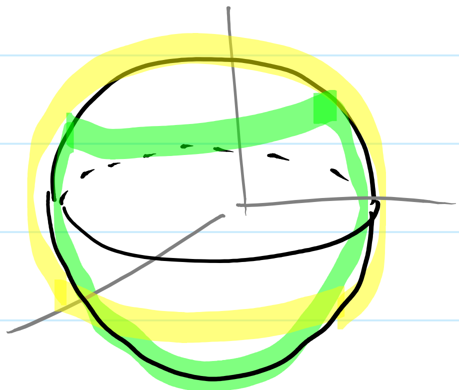
α = a product of loops in A, B, or $A\cap B$

Algebraically, $\pi_1(X) = \pi_1(A) \times_{\pi_1(A\cap B)} \pi_1(B)$

For $S^n = \{ x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : \|x\| = 1 \}$

Let $A = \{ x \in S^n : x_{n+1} > \frac{-1}{2} \}$

$B = \{ x \in S^n : x_{n+1} < \frac{1}{2} \}$



Both A, B are homeomorphic to $D^n = \{ y \in \mathbb{R}^n : \|y\| \leq 1 \}$

There A, B are contractible

For $n \geq 2$, $A\cap B$ is path connected

not true if $n=1$

Therefore, $\pi_1(S^n) = 1$ if $n \geq 2$

As a matter of fact, $\pi_1(S^1)$ is not trivial and it will be discussed.